

### 4.1.3. Applications of Green's Function Method.

\* Basic steps:

- ① Determine the appropriate homogeneous problem related to the original nonhomogeneous heat conduction problem.  
(Let  $g=0$  and all B.Cs homogeneous)
- ② Find the solution of the appropriate homogeneous problem, by the method of separation of variables, for example (in most cases)
- ③ Determine the Green's function at  $\tau=0$ ,  $G(\vec{r}, t | \vec{r}', 0)$ , from the solution of the homogeneous problem; and make the Green's function  $G(\vec{r}, t | \vec{r}', \tau)$  by replacing "t" in  $G(\vec{r}, t | \vec{r}', 0)$  by " $t-\tau$ ".
- ④ Determine the solution of the original nonhomogeneous transient conduction problem using the general expression of  $T(\vec{r}, t)$  in terms of  $G(\vec{r}, t | \vec{r}', \tau)$ .

\* Example 1.

An infinite medium ( $-\infty < x < \infty$ ) is initially at temperature  $F(x)$ , for times  $t > 0$  there is heat generation inside the medium  $g(x, t)$ . Obtain an expression for the temperature distribution  $T(x, t)$  for times  $t > 0$ .

The complete problem: (1D transient conduction)

$$\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{1}{k} g(x,t) = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \quad (-\infty < x < \infty)$$

B.C.  $T|_{x=\pm\infty} = \text{finite}$  (Not really a B.C.)

I.C.  $T|_{t=0} = F(x)$

This is a problem with nonhomogeneous conduction equation.

① Determine the appropriate homogeneous problem:

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \psi(x,t)}{\partial t}$$

B.C.  $T|_{x=\pm\infty} = \text{finite}$

I.C.  $T|_{t=0} = F(x)$

② Find the solution of the appropriate homogeneous problem.

① Separation of  $\psi(x,t)$ .

$$\text{Assume: } \psi(x,t) = X(x) \Gamma(t)$$

then:  $\frac{X''}{X} = \underbrace{\frac{1}{\alpha} \frac{d\Gamma}{dt}}_{\Gamma(t) = C e^{-\beta^2 \alpha t}} = -\beta^2$  ( $\beta \geq 0$ )

② Solving ODE.

and for  $X(x)$ ,

$$\underbrace{X''(x) + \beta^2 X(x)}_0 = 0$$

general solution,

$$\underbrace{X_\beta(x)}_{A(\beta) \cos \beta x + B(\beta) \sin \beta x}$$

There is no restriction for  $\beta$  from boundary conditions,  
so  $\beta$  can assume all values from zero to infinity continuously.

③ Making final solution:

The general solution for  $\psi(x,t)$  is constructed by the superposition of all the elemental solutions by integrating over the value for  $\beta$  from zero to infinity:

$$\psi(x,t) = \underbrace{\int_{\beta=0}^{\infty} [A(\beta) \cos \beta x + B(\beta) \sin \beta x] e^{-\beta^2 t} d\beta}_{\sim}$$

④ Determining unknown coefficient.

Applying initial condition:  $\psi|_{t=0} = F(x)$

$$\text{so: } F(x) = \int_{\beta=0}^{\infty} [A(\beta) \cos \beta x + B(\beta) \sin \beta x] d\beta$$

Using the orthogonal properties (Fourier transformation).

We obtain:

$$\begin{cases} A(\beta) = \frac{1}{\pi} \int_{x'=-\infty}^{+\infty} F(x') \cos \beta x' dx \\ B(\beta) = \frac{1}{\pi} \int_{x'=-\infty}^{+\infty} F(x') \sin \beta x' dx \end{cases}$$

And:

$$A(\beta) \cos \beta x + B(\beta) \sin \beta x = \frac{1}{\pi} \int_{x'=-\infty}^{+\infty} F(x') \cos \beta(x-x') dx'$$

Therefore:

$$\psi(x,t) = \underbrace{\frac{1}{\pi} \int_{\beta=0}^{\infty} e^{-\beta^2 t} \int_{x'=-\infty}^{+\infty} F(x') \cos \beta(x-x') dx' d\beta}_{\sim}$$

Note:

$$\int_{\beta=0}^{\infty} e^{-\beta^2 t} \cos \beta(x-x') d\beta = \sqrt{\frac{\pi}{4\pi t}} e^{-\frac{(x-x')^2}{4at}}$$

$$\text{so: } \psi(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{x'=-\infty}^{+\infty} F(x') e^{-\frac{(x-x')^2}{4\alpha t}} dx'$$

③ Determine the Green's function.

The solution of the appropriate homogeneous problem can be written as:

$$\psi(x,t) = \int_{x'=-\infty}^{+\infty} \left[ \frac{1}{\sqrt{4\pi\alpha t}} e^{-\frac{(x-x')^2}{4\alpha t}} \right] F(x') dx'$$

In terms of Green's function (for  $\tau=0$ ):

$$\psi(x,t) = \int_{x'=-\infty}^{+\infty} \underbrace{G(x,t|x',\tau)}_{\tau=0} F(x') dx'$$

Therefore:

$$G(x,t|x',\tau) \Big|_{\tau=0} = \frac{1}{\sqrt{4\pi\alpha t}} e^{-\frac{(x-x')^2}{4\alpha t}}$$

and the Green's function for any  $\tau$  becomes:

$$G(x,t|x',\tau) = \frac{1}{\sqrt{4\pi\alpha(t-\tau)}} e^{-\frac{(x-x')^2}{4\alpha(t-\tau)}}$$

④ Determine the solution of the original nonhomogeneous problem.

$$T(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{x'=-\infty}^{+\infty} e^{-\frac{(x-x')^2}{4\alpha t}} \bar{F}(x') dx'$$

$$+ \frac{\alpha}{K} \int_{\tau=0}^t d\tau \int_{x'=-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\alpha(t-\tau)}} e^{-\frac{(x-x')^2}{4\alpha(t-\tau)}} g(x',\tau) dx'$$

\* Example 2.

A cylinder  $0 \leq r \leq b$  is initially at temperature  $F(r)$ . For times  $t > 0$ , there is heat generation at a rate of  $g(r, t)$  while the boundary surface at  $r = b$  is kept at temperature  $f(t)$ . Obtain an expression for the temperature distribution  $T(r, t)$ .

The complete problem:  $T = T(r, t)$

$$\boxed{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{k} g(r, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}}$$

B.C.  $\begin{cases} T|_{r=0} = \text{finite} \\ T|_{r=b} = f(t) \end{cases}$

I.C.  $T|_{t=0} = F(r)$

Solution through Green's function method.

① Determine the related homogeneous problem.

$$\boxed{\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{1}{\alpha} \frac{\partial \psi}{\partial t}}$$

B.C.  $\begin{cases} \psi|_{r=0} = \text{finite} \\ \psi|_{r=b} = 0 \quad \leftarrow \text{homogeneous} \end{cases}$

I.C.  $\psi|_{t=0} = F(r)$

② Find the solution of the related homogeneous problem.

$$\psi(r, t) = \int_{r'=0}^b \left[ \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{J_0(\beta_m r)}{J_0^2(\beta_m b)} J_0(\beta_m r') \right] F(r') r' dr'$$

with  $\beta_m$  determined by:  $J_0(\beta_m b) = 0$  (eigenvalues)

③ Determine the Green's function at  $\tau=0$  and arbitrary  $\tau$ .

$$\text{As } \psi(r, t) = \int_{r'=0}^b G(r, t | r', \tau) F(r') r' dr'$$

$$\text{so: } G(r, t | r', \tau) \Big|_{\tau=0} = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{J_0(\beta_m r)}{J_1^2(\beta_m b)} J_0(\beta_m r')$$

replacing  $t$  by  $t-\tau$ :

$$G(r, t | r', \tau) = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 (t-\tau)} \frac{J_0(\beta_m r)}{J_1^2(\beta_m b)} J_0(\beta_m r')$$

④ Determine the solution of original nonhomogeneous problem.

$$T(r, t) = \int_{r'=0}^b G(r, t | r', \tau) \Big|_{\tau=0} F(r') r' dr' + \frac{2}{k} \int_{\tau=0}^t \int_{r'=0}^b G(r, t | r', \tau) g(r', \tau) r' dr' d\tau \\ - \alpha \int_{\tau=0}^t \left[ r' \frac{\partial G}{\partial r'} \right]_{r'=b} f(\tau) d\tau$$

$$\text{Note: } r' \frac{\partial G}{\partial r'} \Big|_{r'=b} = - \frac{2}{b} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 (t-\tau)} \beta_m \frac{J_0(\beta_m r)}{J_1(\beta_m b)}$$

therefore,

$$T(r, t) = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{J_0(\beta_m r)}{J_1^2(\beta_m b)} \int_{r'=0}^b J_0(\beta_m r') F(r') r' dr' \\ + \frac{2\alpha}{kb^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{J_0(\beta_m r)}{J_1^2(\beta_m b)} \int_{\tau=0}^t e^{\alpha \beta_m^2 \tau} d\tau \int_{r'=0}^b J_0(\beta_m r') g(r', \tau) r' dr' \\ + \frac{2\alpha}{b} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \beta_m \frac{J_0(\beta_m r)}{J_1(\beta_m b)} \int_{\tau=0}^t e^{\alpha \beta_m^2 \tau} f(\tau) d\tau$$

\* Example 3.

A solid sphere of radius  $r=b$  is initially at temperature  $F(r, \xi)$ . For times  $t > 0$  heat is generated at a rate  $g(r, \xi, t)$ , while the boundary surface is kept at zero temperature. Obtain an expression for the temperature distribution  $T(r, \xi, t)$  in the sphere.

The complete problem: ( $\xi = \cos \theta$ )

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \xi^2} \left[ (1-\xi^2) \frac{\partial T}{\partial \xi} \right] + \frac{1}{k} g(r, \xi, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$\text{B.C. } \left. T \right|_{r=0} = \text{finite}$$

$$\left. T \right|_{r=b} = 0$$

$$\text{I.C. } \left. T \right|_{t=0} = F(r, \xi)$$

Solution by Green's function method.

① Determine the appropriate homogeneous problem.

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \xi^2} \left[ (1-\xi^2) \frac{\partial \psi}{\partial \xi} \right] = \frac{1}{\alpha} \frac{\partial \psi}{\partial t}$$

$$\text{B.C. } \left. \psi \right|_{r=0} = \text{finite}$$

$$\left. \psi \right|_{r=b} = 0$$

$$\text{I.C. } \left. \psi \right|_{t=0} = F(r, \xi)$$

② Find the solution of the appropriate homogeneous problem

$$\begin{aligned} \psi(r, \xi, t) = & \int_{r'=0}^b \int_{\xi'=-1}^{+1} \left[ \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{N(n) N(\lambda_{nm})} e^{-\alpha \lambda_{nm}^2 t} \cdot (r/r')^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda_{nm} r') P_n(\xi') J_{n+\frac{1}{2}}(\lambda_{nm} r') P_n(\xi') \right] \\ & \cdot F(r', \xi') r'^2 d\xi' dr' \end{aligned}$$

$$\text{Where: } \begin{cases} N(n) = \frac{2}{2n+1} \\ N(\lambda_{nm}) = -\frac{b^2}{2} J_{n-\frac{1}{2}}(\lambda_{nm} b) J_{n+\frac{3}{2}}(\lambda_{nm} b) \end{cases}$$

③ Determine the Green's function.

$$\text{As } \Psi(r, \xi, t) = \int_{r'=0}^b \int_{\xi'=-1}^{+1} G(r, \xi, t | r', \xi', \tau) \Big|_{\tau=0} F(r', \xi') r'^2 d\xi' dr'$$

$$\text{So: } G(r, \xi, t | r', \xi', \tau) \Big|_{\tau=0} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{N(n) N(\lambda_{nm})} e^{-\alpha \lambda_{nm}^2 t} (r' r)^{-\frac{1}{2}} \int_{n+\frac{1}{2}}^{n+1} (\lambda_{nm} r) P_n(\xi) J_{n+\frac{1}{2}}(\lambda_{nm} r') P_n(\xi')$$

Replacing  $t$  by  $t-\tau$ :

$$G(r, \xi, t | r', \xi', \tau) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{N(n) N(\lambda_{nm})} e^{-\alpha \lambda_{nm}^2 (t-\tau)} (r' r)^{-\frac{1}{2}} \int_{n+\frac{1}{2}}^{n+1} (\lambda_{nm} r) P_n(\xi) J_{n+\frac{1}{2}}(\lambda_{nm} r') P_n(\xi')$$

④ Determine the solution of the original problem.

$$T(r, \xi, t) = \int_{r'=0}^b \int_{\xi'=-1}^{+1} G(r, \xi, t | r', \xi', \tau) \Big|_{\tau=0} F(r', \xi') r'^2 d\xi' dr' + \frac{\alpha}{K} \int_{\tau=0}^t d\tau \int_{r'=0}^b \int_{\xi'=-1}^{+1} G(r, \xi, t | r', \xi', \tau) g(r', \xi', \tau) r'^2 d\xi' dr'$$

Therefore:

(Note: B.C. is homogeneous)

$$T(r, \xi, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{N(n) N(\lambda_{nm})} e^{-\alpha \lambda_{nm}^2 t} r^{-\frac{1}{2}} \int_{n+\frac{1}{2}}^{n+1} (\lambda_{nm} r) P_n(\xi) \int_{r'=0}^b \int_{\xi'=-1}^{+1} r'^{\frac{3}{2}} J_{n+\frac{1}{2}}(\lambda_{nm} r') P_n(\xi') F(r', \xi') d\xi' dr' + \frac{\alpha}{K} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{N(n) N(\lambda_{nm})} e^{-\alpha \lambda_{nm}^2 t} r^{-\frac{1}{2}} \int_{n+\frac{1}{2}}^{n+1} (\lambda_{nm} r) P_n(\xi) \int_{\tau=0}^t e^{\alpha \lambda_{nm}^2 \tau} d\tau \int_{r'=0}^b \int_{\xi'=-1}^{+1} r'^{\frac{3}{2}} J_{n+\frac{1}{2}}(\lambda_{nm} r') P_n(\xi') g(r', \xi', \tau) d\xi' dr'$$